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On the extreme of internal entropy production

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Abstract
Using the one-dimensional diffusion equation the extrema of internal entropy production are investigated. It is shown that the internal entropy production is essentially a convex function, with only a minimum in Euler variation with respect to temperature, regardless of if the system is open or closed. The maximum internal entropy production can only occur in variation with respect to a physical coefficient. It is found that either a maximum or a minimum internal entropy production can occur with variation with respect to a diffusivity coefficient. The maximum tends to occur for the closed boundary condition while the minimum tends to occur for the open boundary condition. For a non-steady state, the study indicates that the extreme results shown in the steady state are not guaranteed to occur under the non-steady condition. The work highlights that the physics behind determining extrema of internal entropy production is still not fully understood.

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1. Introduction

Often an extremum principle is crucial in determining the most appropriate state of a system. Generally, even with physical restrictions on energy, mass, momentum, etc, a system can often still have a variety of possible solutions.

In 1975, Paltridge [1] introduced an extremum principle in a simple climate model corresponding to the maximization of the atmospheric internal entropy production due to climate processes, by which the meridional distribution of temperature, cloud-cover and meridional energy flux can reasonably be matched with observations. Since then, a number of works [2–18, 20] have addressed the maximum internal entropy production (MaxIEP, or simply MaxEP or MEP) problem in atmospheric/oceanic science, life science and physics.

For a long time, the fundamental physical basis for MaxIEP has been a puzzle. The internal entropy production pertains to dissipation, so why does nature always follow the maximum dissipation path to equilibrate a system?

Even in earlier decades, the work [1] was challenged by Rodgers [21], as he pointed out there was not any theoretical justification for MaxIEP and also a similar result of [1] could be obtained by minimizing the internal entropy production with respect to temperature. The minimum internal entropy production was usually referred to as the Prigogine principle [22]. Rodgers also questioned whether MaxIEP can be applied to other planets since those planets might not have enough atmosphere or ocean to carry the meridional conducting heat flow, which is essential for applying MaxIEP. MaxIEP was not applied to other planets until many years later [10]; however, lately it has been argued that the results for other planets obtained based on MaxIEP do not seem consistent with the results obtained by variation with respect to temperature [23]. The approach to studying extrema of internal entropy production appears always controversial. Recently, it has been claimed [24, 25] that MaxIEP has been approved based on statistic physics; however, the proof is not convincing. For example, errors in the derivations for [25] have been noted [26]. It is also noted [23] that a statistical mechanics approach [24] is unconstrained, apart from energy and mass conservation. It is emphasized in [24, 25] that the proof is based on Jaynes' principle. Jaynes [27] used an inverse statistical mechanics approach to obtaining the probability density from a set of events which are discrete and mutually exclusive. For such a process, Jaynes' approach makes maximization of entropy subject to the constraints on the set of events. Dewar [24] applied Jaynes' principle constraining the continuous macrophysical variables of the system. It is questionable in physics and difficult in technique to embed continuous macrophysical variables with infinite freedom (like Earth's climate system) to a finite discrete system. This is a critical point which makes any proof of MaxIEP based on Jaynes' principle hard to be justified, even though Jaynes' theory has itself aroused much attention (for example, see, [28]).

A number of questions related to MaxIEP still lack clear answers. Is MaxIEP a general rule or only conditionally true? How do we obtain MaxIEP, through Euler variation with respect to temperature or variation with respect to other physical parameters? Most discussions of MaxIEP are restricted to steady-state systems; can MaxIEP be extended to non-steady state systems? An examination of these questions is the purpose of this paper. If MaxIEP is a true principle, it should be applicable to any system. In the same way as Paltridge [1], we choose a one-dimensional system to illustrate the problem.

2. Euler variation and minimum internal entropy production

Consider the diffusion equation for thermal conduction in a one-dimensional domain L with the boundary ∂L,

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}j(x,t) + \gamma u(x,t) - f(x,t) = 0 \quad (x \in L), \tag{1}$$

where $u(x,t)$ is temperature, $j(x,t)$ is the thermal conducting flow, $f(x,t)$ is a positive definite source term and γu represents the sink term with γ a constant. By Fourier's law the thermal conducting flow $j = -k\partial u/\partial x$, where k is the thermal conducting coefficient. For simplicity, we neglect the specific heat c_p in (1), and let all variables in (1) have arbitrary units.

Equation (1) is equivalent to the governing equation in an energy balance model [29] except that the Earth's curvature effect is neglected. The Earth's curvature effect can be included simply by replacing the derivative with the covariant (counter-covariant) derivative [30]. The one-dimensional energy balance model describes the zonal averaged climate from the pole to the equator, which is equivalent to the one-dimensional box model used in [1]. In an energy balance model, usually $u(x,t) = A + BT$ is set, where A and B are constants, and T is the temperature, A can be moved into $f(x,t)$, without loss of generality, and $f(x,t)$ and γu correspond to the incoming solar and outgoing infrared energy respectively.

Based on (1) the entropy balance equation is

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{1}{u}\frac{\partial u}{\partial t} \\ &= -\frac{\partial}{\partial x}\left(\frac{j}{u}\right) + j\frac{\partial}{\partial x}\left(\frac{1}{u}\right) - \left(\gamma - \frac{f}{u}\right),\end{aligned} \tag{2}$$

where s is the entropy. In the last equation of (2) the first term is the convergence of entropy flux j/u, the second term is the internal entropy production and the last term represents the entropy production related to the source/sink term.

For the steady-state system, with Fourier's law (1) becomes

$$ku(x)'' - \gamma u(x) + f(x) = 0, \tag{3}$$

where the prime represents $\partial/\partial x$. The internal entropy production is

$$\sigma = \int_L j\frac{\partial}{\partial x}\left(\frac{1}{u}\right) = \int_L k\frac{u'^2}{u^2}. \tag{4}$$

dx in the integration is neglected in (4), the integration variable x could be indicated by the domain L.

Euler variation with respect to u is used to find the extreme value of σ. First, we discuss a system with Dirichlet boundary condition (BC) $\delta u = 0$ $(x \in \partial L)$, where δ represents the variation. By using (A.4) we obtain

$$\frac{1}{u}\left(\frac{u'}{u}\right)' = 0. \tag{5}$$

Since $1/u$ cannot be zero,

$$\frac{u'}{u} = c, \tag{6}$$

where c is the integral constant determined by the BC. If $u = u_{1,2}$ at the boundary points of $x_{1,2} \in \partial L$, $c = \ln(u_2/u_1)/l$ is derived, where $l = x_2 - x_1$, and hence

$$\sigma = \frac{k}{l}\ln^2\left(\frac{u_2}{u_1}\right). \tag{7}$$

It can be shown that the extremum (7) corresponds to a minimum state of internal entropy production. Assuming an arbitrary real function $h(x) \in C^1$ satisfying the same Dirichlet BC $h(x_{1,2}) = u_{1,2}$, by the Cauchy–Schwartz inequality we have

$$\int_L k\frac{h(x)'^2}{h(x)^2} \geqslant \frac{1}{l}\left[\int_L \sqrt{k}\frac{h(x)'}{h(x)}\right]^2 = \frac{k}{l}\ln^2\left(\frac{u_2}{u_1}\right). \tag{8}$$

Therefore variation with respect to temperature only yields the minimum state of internal entropy production.

By Fourier's law the heat flow and the temperature are linearly related. Moreover, generally we can assume the heat flow following the Onsager relation [31]

$$j(x) = \chi\frac{\partial}{\partial x}\frac{1}{u},$$

where χ is the Onsager coefficient, and now the flow associates with temperature nonlinearly. If $\chi = ku^2$, the result becomes the same as above. More generally it is set to be $\chi = ku^n$ [32], where n is an arbitrary number. We have

$$\sigma = \int_L \chi \left(\frac{u'}{u^2}\right)^2 = \int_L ku^{n-4}u'^2. \tag{9}$$

Following the similar procedure of (5)–(8), we can prove that the result is the same as the Euler variation under the Dirichlet BC only leads to the minimum internal entropy production.

Why does Euler variation with respect to temperature only lead to the minimum internal entropy production state? The reason is the convexity of internal entropy production. Generally for a function $f(x, u(x))$, if $\partial^2 f(x, u(x))/\partial u^2 > 0$, then $f(x, u(x))$ is a strong convex function and obviously $1/u^2$ is a strong convex function. The product of a strong convex function with a positive continuous function is again a strong convex function in the same domain [33, 34]. Therefore u'^2/u^2 is a strong convex function as well. The same is true for $u^{n-4}u'^2$. Furthermore, if $f(x, u(x))$ is a strongly convex function, $\int_L f(x, u(x))$ is a convex integral function, and it is only possible for a minimum to exist for a convex integral function.

Under the minimum condition of (6) it is derived that $u'' = c^2u$, and by using (3) we have

$$\frac{u'}{u} = \frac{f' - \gamma u'}{f - \gamma u} = c, \qquad \delta u = 0\ (x \in \partial L). \tag{10}$$

Therefore the minimum internal entropy production is essentially determined by the relationship between the source and sink terms.

If the system is considered with the Neumann BC $\delta u' = 0$ $(x \in \partial L)$, the corresponding variation becomes more complicated (see appendix A). In [3] the variation method was applied to obtain MaxIEP in a particular Neumann BC, $u' = 0$ $(x \in \partial L)$, we call such a BC closed BC. The other Neumann BC can be called open BC.

Since under the closed BC, $u' = 0$ $(x \in \partial L)$,

$$\begin{aligned}\sigma &= \int_L k\frac{u'^2}{u^2} \\ &= -\int_L k\frac{\partial}{\partial x}\left(\frac{u'}{u}\right) + \int_L k\frac{u''}{u} \\ &= \int_L k\frac{u''}{u}\end{aligned} \tag{11}$$

substituting (3) into (11), the internal entropy production becomes

$$\sigma = \int_L \frac{k\gamma u''}{ku'' + f}. \tag{12}$$

Following Euler variation with the Neumann BC (A.6), we have

$$\frac{\partial}{\partial x}\frac{\partial}{\partial u''}\frac{k\gamma u''}{ku'' + f} = \frac{\partial}{\partial x}\frac{\gamma kf}{(ku'' + f)^2} = 0. \tag{13}$$

Equation (13) yields

$$ku'' = Cf^{\frac{1}{2}} - f, \tag{14}$$

by using $u' = 0 \in \partial L$, it is derived $C = \int_L f \big/ \int_L f^{\frac{1}{2}}$. From (3) and (14)

$$u = \frac{1}{\gamma}Cf^{\frac{1}{2}}. \tag{15}$$

The above result in (14)–(15) is the same as that in [3], but obtained from a different approach with a rigorous mathematical basis (appendix A).

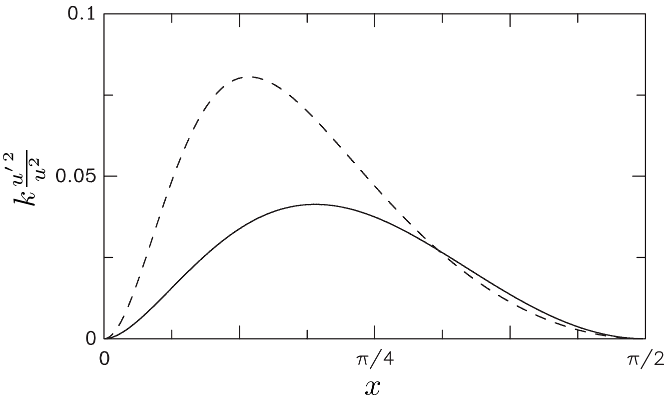

Figure 1. The distributions of internal entropy production, $k\frac{u'^2}{u^2}$, based on the regular solution of (B.2) (dashed lines), and based on (16) (solid lines). Physical quantities are in arbitrary units.

Figure 1 shows a distribution of the internal entropy production inside L based on (11) with an extreme solution of (14)–(15), i.e. the distribution of

$$k\frac{u'^2}{u^2} = k\left[\frac{\gamma\int_{x_1}^{x}\left(Cf^{\frac{1}{2}} - f\right)}{kCf^{\frac{1}{2}}}\right]^2. \tag{16}$$

We assume the domain $L = [0, \pi/2]$ with the source function $f(x) = \sin(x)$ which is similar to the distribution of incoming solar energy source from the northern pole ($x = 0$) to the equator ($x = \pi/2$). It is found that the result is not sensitive to γ. However generally $\gamma\langle u\rangle \sim \langle f\rangle$ should hold, where the bracket means a domain average, otherwise the heat will be accumulated. We let $\gamma = 0.5$ make u close to the order of unit, since $\langle f\rangle \sim 0.5$. The closed BC is ensured by assuming that there is no conducting heat flow across the equator and the pole. Figure 1 shows that the BC of $u' = 0$ ($x \in \partial L$) is satisfied as $u' = \frac{1}{k}\int_{x_1}^{x}\left(Cf^{\frac{1}{2}} - f\right)$ with C defined above.

Also the distribution of internal entropy production based on the regular solution (see appendix B with Neumann BC) is shown. It is found that the internal entropy production, σ, is much higher based on the regular solution of (B.2) in comparison with the optimized result of (16). It was claimed [3] that the extreme of internal entropy production corresponds to MaxIEP. Such a statement was incorrect. The result shown in figure 1 further confirms that variations with respect to temperature can only yield the minimum internal entropy production, whether the boundary is Dirichlet or Neumann. It is worth pointing out that the minimum of internal entropy production through variation of temperature was investigated before [19, 21, 23]. Also the minimum internal entropy production was confirmed in [21].

Of course, we are not interested in the case without source/sink thus where the steady equation (3) becomes $u'' = 0$, which only leads the linear distribution of u in L under Dirichlet BC or $u \equiv 0$ under closed BC. The above Euler variation does not exist. Equations (10) and (16) show that the optimized results are always associated with the source/sink terms.

3. Extremes in physical coefficient space

In the previous section it is shown generally that minimum optimization exists only for internal entropy production because of its convex nature. However, Paltridge and others have demonstrated that MaxIEP indeed exists in simple climate models. How do we understand this conflict between the minimum optimization of the Euler variation and MaxIEP. In Paltridge's model, MaxIEP was applied through the adjustment of the distribution of physical quantities, especially the distribution of cloud, which affects the distribution of the incoming/outgoing radiation energy. Consequently, the poleward transport conducting heat flow is affected. In other words, the optimization was achieved through the change of the physics inside the system. In the simple system of (3), such a change corresponds to the variation with respect to the thermal conduction coefficient k. This was first done correctly by [5] and later followed by others [10, 12, 15]. The variation in k can be completely justified physically, as it is essentially equivalent to the optimization through the meridional conducting heat flow. It is worthy of mention that the MaxIEP principle has been applied to the oceanic turbulence [35–37]. Also the seeking of extremes is accomplished by variation with respect to the physical variables, such as surface torque.

However under which condition does MaxIEP hold true is not clear even for the simple one-dimensional system. This will be explored further. The internal entropy production can be generally written as

$$\sigma(k) = \int_L k\frac{u'^2}{u^2} = \int_L F(k,u,u'), \tag{17}$$

where u and u' should also be the function of k. Considering the neighboring curves of $u(x,k) + \delta k\eta(x)$, where $\eta(x)$ is a real function, we define the second variation of functional (17) as the term in the expansion of $\sigma(k)$, in which the powers contain δk^2 [34], i.e. we put

$$\begin{aligned}\delta^2\sigma(k) &= \frac{1}{2}\delta k^2\left[\frac{\mathrm{d}^2\sigma(k)}{\mathrm{d}k^2}\right]\bigg|_{dk=0}\\ &= \frac{1}{2}\delta k^2\int_L \frac{\mathrm{d}^2F}{\mathrm{d}k^2}\eta(x)^2\\ &= \frac{1}{2}\delta k^2\int_L\left[\frac{\partial^2 F}{\partial k^2} + \frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial k}\right)^2 + \frac{\partial^2 F}{\partial u\partial u'}\frac{\partial u}{\partial k}\frac{\partial u'}{\partial k} + \frac{\partial^2 F}{\partial u'^2}\left(\frac{\partial u'}{\partial k}\right)^2\right]\eta^2. \end{aligned}\tag{18}$$

It is difficult to judge the sign of $\delta^2\sigma(k)$ in the general case of (18). However under the closed BC: $u' = 0$ ($x \in \partial L$), (11) and (3) yield

$$\sigma = \int_L k\frac{u''}{u} = \int_L\left(\gamma - \frac{f}{u}\right), \tag{19}$$

consequently (18) becomes

$$\delta^2\sigma(k) = \delta k^2\int_L -2\frac{f}{u^3}\left(\frac{\partial u}{\partial k}\right)^2\eta^2 < 0, \tag{20}$$

since both u and f are positive. Therefore, if there exists an extremum it must be a maximum. We conclude that a system with a closed BC tends to have MaxIEP in the physical k space.

Figure 2 shows the variation of internal entropy production with respect to k for the closed BC as shown in (19). The source/sink function and domains are the same as those in figure 1. The solution of u is obtained by (A.2) for the Neumann BC. It is shown in figure 2 that the internal entropy production indeed has a maximum point at $k \approx 0.07$.

It is now clear why the results obtained in [10] and [23] are different for the application of MaxIEP principle to the Earth and other planets, since the variation was applied to the thermal conducting coefficient in [10] but the variation was applied to temperature in [23].

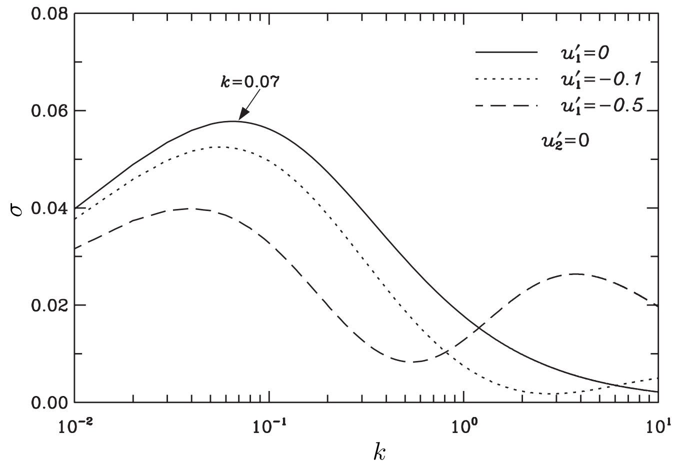

Figure 2. Internal entropy production, σ, in the domain $[0, \frac{\pi}{2}]$ versus the variation of k for a closed BC case ($u_1' = u_2' = 0$, at $x_{1,2} \in \partial L$), and for two other open system cases with leaking thermal flows at the boundary. Again, physical quantities are in arbitrary units.

We will now illustrate the related physics in more detail to understand why MaxIEP occurs in k space. Figure 3 displays the distributions of u, u' and internal entropy production corresponding to $k = 0.02$, 0.07 and 1 for the closed case of (19). For $k = 0.02$, u' is very large due to the small thermal conduction (diffusion) coefficient, but the thermal conducting heat flow $j = -ku'$ is small, due to a small value of k. Therefore the internal entropy production $\int ku'^2/u^2$ cannot be the maximum. For $k = 1$, though, k is large but u' is very small due to strong thermal conduction (diffusion) which smoothes the temperature gradient. The internal entropy production is also small. MaxIEP can only be achieved for an intermediate value of k. Too weak or too strong thermal conduction (diffusion) inhibits an extremum.

The internal entropy production corresponds to the dissipation process. From (4)

$$\sigma \sim -\mathrm{j}\frac{\partial u}{\partial x}$$

the positive internal entropy production indicates that the flow moves along the gradient of temperature from the higher to lower temperature region. During this process the flow tends to smooth out the gradient of the temperature field by extracting energy from the high temperature region. This results in the redistribution of temperature and the destruction of the existing temperature gradient. Therefore the internal entropy production is referred to as dissipation. On the other hand, the negative internal entropy production refers to the flow that moves along counter to the gradient of temperature and thus enhances the slope of the temperature. This process, of course, would not happen unless forced externally.

MaxIEP corresponds to a maximum dissipation process that smoothes out the temperature structure of the system. In our example of (3), the temperature structure is created by the inhomogeneous source/sink from radiation. Equation (2) shows that the entropy production corresponding to the source/sink is

$$-\int_L \left(\gamma - \frac{f}{u}\right),$$

the same as (19) but with opposite sign. Therefore in steady closed BC cases, the source/sink tends to create maximum temperature gradients; whereas the thermal conducting flow tends to

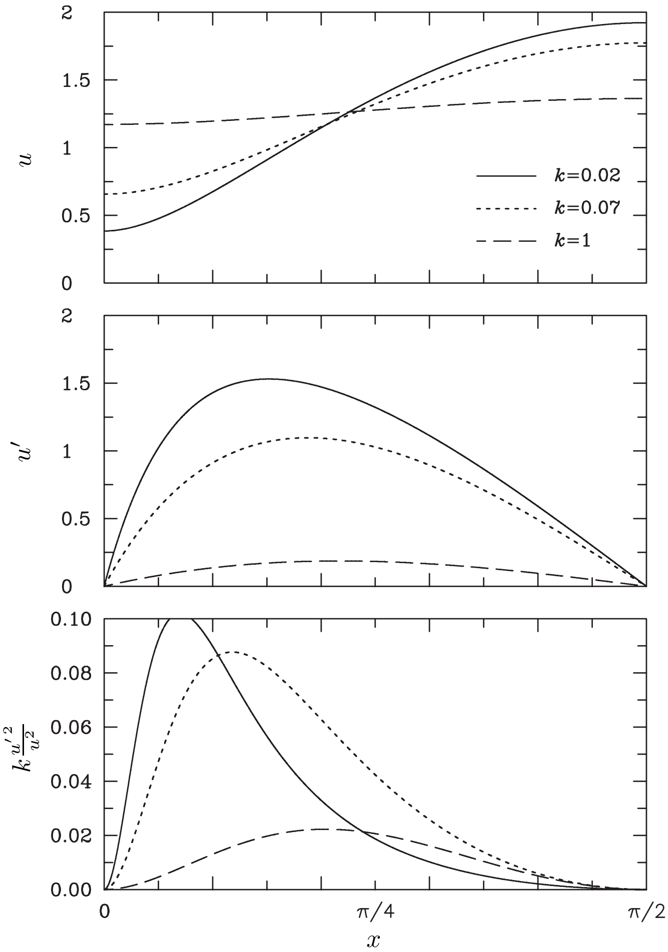

Figure 3. Distributions of u, u' and internal entropy production, $k\frac{u'^2}{u^2}$, in domain $[0, \frac{\pi}{2}]$ for the closed case ($u_1' = u_2' = 0$ at $x_{1,2} \in \partial L$), three cases of $k = 0.02, 0.07$ and 1 are shown. Again, physical quantities are in arbitrary units.

be a maximum at diminishing the temperature gradient (i.e. smearing out the structure). Without the maximum negative entropy production by the source/sink, the maximum dissipation cannot persist for long.

So far, the study is limited to closed BC cases. Does MaxIEP hold true under other BC conditions? In figure 2, besides the result for closed BC, the other Neumann BC cases with thermal flow leaking out at the boundaries are also considered. For a very weak leaking case as $u_1' = -0.1$ (keeping $u_2' = 0$ where $x_{1,2} \in \partial L$) it is shown that the maximum profile structure of internal entropy production for the closed case changes to a wavy structure with both a maximum and a minimum appearing. In this situation the system probably still persists with MaxIEP to some extent. With a large leaking of $u_1' = -0.5$ and $u_2' = 0$, the curve structure becomes further smoother and an obvious minimum region appears. In this situation MaxIEP may no longer be true. Therefore, MaxIEP only exists in the steady closed system or a steady quasi-closed system with a very weak connection exchange with the environment across the boundary.

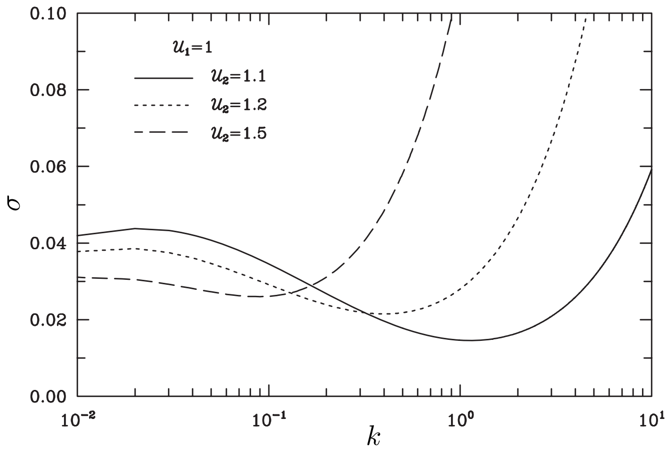

Figure 4. Internal entropy production, σ, in domain [0, 1] versus the variation of k for an open system with Dirichlet BC. Physical quantities are in arbitrary units.

Let us further consider an open system with Dirichlet BC (solution of (B.3) in appendix B). In Dirichlet BC there is no restriction on thermal conducting flow at boundary, therefore such a BC can be referred to as 'free' open BC. The source/sink and domain are the same as those of figure 2. The BC is set as $u_1 = 1$ and $u_2 = 1.1$, 1.2 and 1.5 with $x_{1,2} \in \partial L$. Since the input source, f, increases toward x_2 in the domain of L, it is reasonable to assume $u_2 > u_1$. This is similar to the Earth system that the temperature is higher in the equator than at the polar regions. Figure 4 shows the variation of internal entropy production with respect to k. It is found for $u_2 = 1.1$ that there is a very weak MaxIEP at $k \approx 0.02$ but an obvious minimum internal entropy production at $k \approx 1.0$. Which of the two extremes does nature prefer? As the slope of u increases MaxIEP diminishes and the internal entropy production dramatically increases toward large k, and generally only the minimum exists.

Figure 5 shows the distributions of u, u' and internal entropy production corresponding to $k = 0.02$, 1.13 and 5 for the case of $u_1 = 1$ and $u_2 = 1.1$ shown in figure 4. As mentioned a value of $k = 0.02$ corresponds to MaxIEP and $k = 1.13$ corresponds to the minimum internal entropy production. At $k = 0.02$, the very small thermal conducting coefficient causes dramatic uneven distributions of u and u'. It is hard to believe this MaxIEP state is nature's choice. At $k = 1.13$, the distributions of u and u' become smoother. The very small value of u'^2 produces the minimum internal entropy production. At $k = 5$, the distributions for both u and u' are very smooth, but the large value of k causes a larger internal entropy production compared to the case of $k = 1.13$. Figure 5 helps to understand the appearance of the minimum internal entropy production in the intermediate range of k.

The results shown in figure 4 are also true for other source functions. Figure 6 is similar to figure 4 in physical input but with a source function $f(x) = \alpha x$, where α is a constant. Two values of $\alpha = 0.4$ and 0.6 are considered. It is found that the main feature of internal entropy production is similar to that shown in figure 4. For $\alpha = 0.4$, generally there are small local maxima in the small-k region and large minima in the larger-k region. For $\alpha = 0.6$, the maxima in the small-k region disappear and only the large minima persist.

Therefore in an open (free open) BC system there usually exists a minimum in the internal entropy production which suggests the open system tends to relax by choosing the minimum

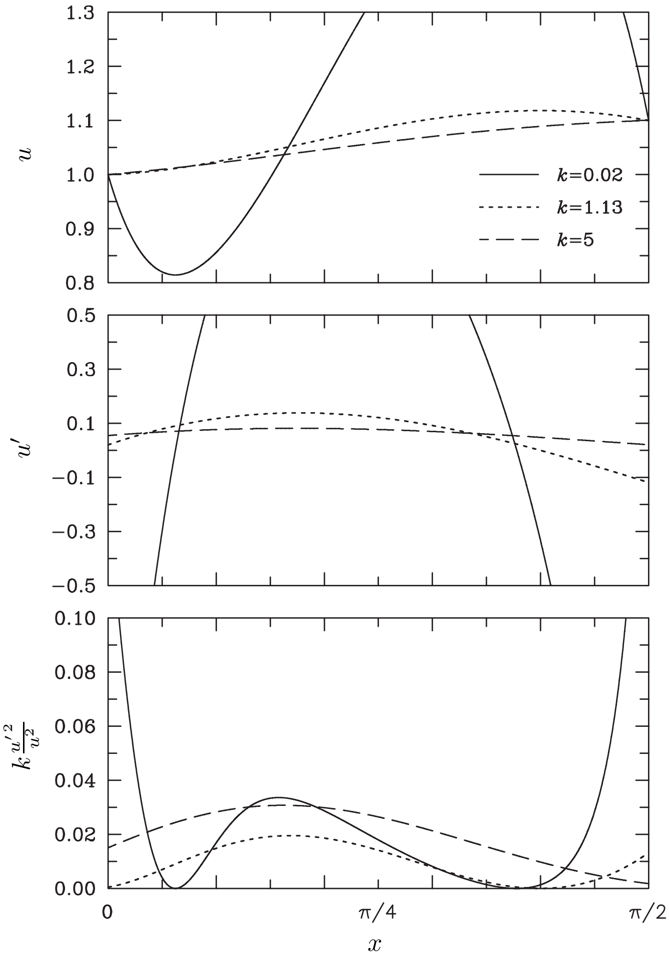

Figure 5. Distributions of u, u' and internal entropy production, ku'^2/u^2, in the domain $[0, \frac{\pi}{2}]$ for the Dirichlet BC ($u_1 = 1, u_2 = 1.1$); three cases of $k = 0.02$, 1.13 and 5 are shown. Physical quantities are in arbitrary units.

dissipation. A further explanation for nature's choice of optimized solution should be based on instability theory.

4. Non-steady solution

So far only the steady-state case has been discussed. Also almost all previous studies are based on the steady-state case (annual mean is equivalent to a steady state). In this section, the study of extrema will be extended to the non-steady case.

From equation (1), by transforming according to $u(x, t) = \mathrm{e}^{-\gamma t}\tilde{u}(x, t)$, we obtain

$$\frac{\partial}{\partial t}\tilde{u}(x, t) - k\frac{\partial^2}{\partial x^2}\tilde{u}(x, t) = f(x, t)\,\mathrm{e}^{\gamma t}. \qquad (21)$$

This is a standard parabolic equation. Since MaxIEP for closed BC is our main interest, we chose Green's function to satisfy the closed BC as $G(x, t; \xi, \tau)$ in the domain $L = [0, l]$, with

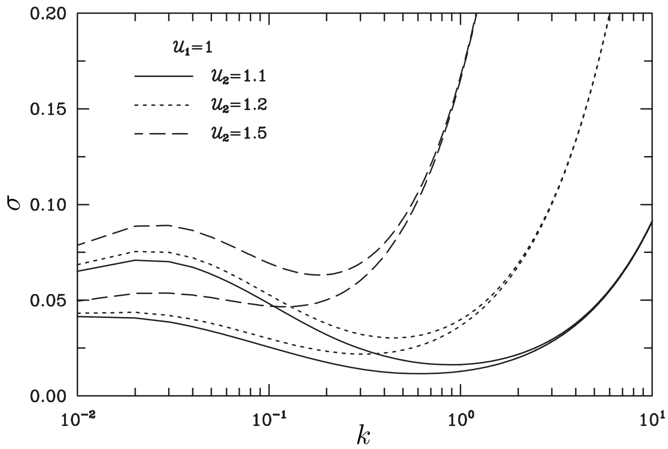

Figure 6. Internal entropy production, σ, in domain [0, 1] versus the variation of k for the system with free open BC. For each value of u_2, the upper (lower) curve corresponding to $\alpha = 0.4$ ($\alpha = 0.6$). Physical quantities are in arbitrary units.

$\frac{\partial G}{\partial x}\big|_{x=0} = \frac{\partial G}{\partial x}\big|_{x=l} = 0$. It is easy to obtain

$$G(x,t;\xi,\tau) = \frac{2}{l}\sum_{n=0}\frac{1}{\delta_n}\mathrm{e}^{-\frac{n^2\pi^2 k(t-\tau)}{l^2}}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\xi}{l}\right), \tag{22}$$

where

$$\delta_n = \begin{cases} 2 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n > 0. \end{cases}$$

For simplicity the zero initial condition $u(x, t = 0) = 0$ is set without loss of generality. The solution of (21) is

$$u(x,t) = \mathrm{e}^{-\gamma t}\int_{\tau=0}^{t}\int_{\xi=0}^{l} f(\xi,\tau)\,\mathrm{e}^{\gamma\tau} G(x,t;\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{23}$$

If the source term is not time dependent with $f(x,t) = f(x)$, we obtain

$$u(x,t) = \frac{2}{l}\sum_{n=0}\frac{A_n}{\delta_n}\frac{1-\mathrm{e}^{-b_n t}}{b_n}\cos\left(\frac{n\pi x}{l}\right), \tag{24}$$

where $b_n = n^2\pi^2 k/l^2 + \gamma$, and

$$A_n = \int_0^l f(\xi)\cos\left(\frac{n\pi\xi}{l}\right)\mathrm{d}\xi.$$

(24) shows $u(x,t)$ approaches a steady state as $t \to \infty$. Also from (24) we derive

$$\frac{\partial}{\partial t}u(x,t) = \frac{2}{l}\sum_{n=0}\frac{A_n}{\delta_n}\mathrm{e}^{-b_n t}\cos\left(\frac{n\pi x}{l}\right). \tag{25}$$

As $t \to \infty$,

$$\frac{\partial}{\partial t}u(x,t) \to 0,$$

which implies the total entropy production is zero in the steady state, since

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{u}\frac{\partial u}{\partial t} = 0.$$

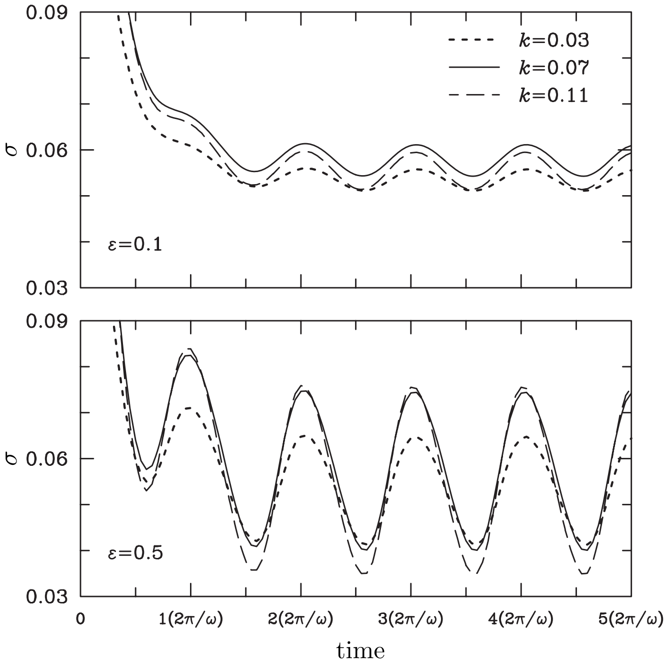

Figure 7. The time evolution of internal entropy production, σ, with three values of k. The other physical inputs are the same as figure 2 with closed BC. Physical quantities are in arbitrary units.

In this situation, the internal entropy production due to thermal conduction flow is counteracted by negative entropy production corresponding to the source/sink.

Consider a case with periodic variation, for simplicity the source function is assumed as

$$f(x)[1+\epsilon\cos(\omega t)],$$

where ϵ is a positive constant. The corresponding solution is

$$u(x,t)=\frac{2}{l}\sum_{n=0}\frac{A_n}{\delta_n}\left\{\frac{1-\mathrm{e}^{-b_n t}}{b_n}+\epsilon\frac{b_n[\cos(\omega t)-\mathrm{e}^{-b_n t}]+\omega\sin(\omega t)}{b_n^2+\omega^2}\right\}\cos\left(\frac{n\pi x}{l}\right). \qquad (26)$$

By (26) the total entropy production $\frac{1}{u}\frac{\partial u}{\partial t}\neq 0$, since for this circumstance the positive internal entropy production by the thermal conducting flow and the negative entropy production by the source/sink cannot reach their positive and negative maximum values at the same time, as they can for the steady case.

Figure 7 shows the time evolution of internal entropy production, $\sigma = \int_0^l k\frac{u'^2}{u^2}$, by (26) with two values of ϵ. In order to compare with the MaxIEP result shown in figure 2, we use the same source/sink and domain zone as those in figure 2. Since the initial condition is set to $u(x,t)|_{t=0} = 0$, when t is small, the result is very different from that of the later stage. However the solutions quickly reach their stable periodic behavior. Three curves are plotted corresponding to $k = 0.02$, 0.07 and 0.2. For $\epsilon = 0.1$ the result of $k = 0.07$ is always larger than those of the other two curves, the same as in figure 2. However for $\epsilon = 0.5$, though $k = 0.07$ still produces the largest internal entropy production on time average, but at any time the result for $k = 0.07$ is not always the maximum. Therefore even for a closed system, it is more reliable to achieve MaxIEP in a steady state or at least in a time-averaged state.

5. Conclusion and outlook

MaxIEP has been investigated using a one-dimensional diffusion equation (equivalent to the energy balance equation). It is found that the internal entropy production is a convex function, with only minimum extrema through variations with respect to temperature, regardless of whether the system is open or closed. It is shown that MaxIEP can only be achieved through variation with respect to the physical coefficient. The result of (20) robustly confirms that MaxIEP indeed exists under closed BC. However under open or free open BC, generally the system exhibits a minimum extremum in the internal entropy production even under variations with respect to the physical coefficients.

MaxIEP study in this area is mostly limited to one-dimensional cases. Paltridge [2] extended his model from one dimensional to two dimensional and found MaxIEP is still true.

The extrema of internal entropy production could be a powerful tool. It has been shown by a number of works that MaxIEP is an effective criterion to find preferred states of simple climate models. In modeling nature the open (or free open) BC is more often considered than the closed BC. For example, the atmospheric convection is a free open boundary process [38]. Though it is shown in this paper that there is a tendency toward minimum internal entropy production for open (free open) BC, and we do not yet know the physics behind such minimizations. We do not know if the minimum solution is really preferred by nature. Otherwise such knowledge could help us to reduce the uncertainty in determining the strength of atmospheric convection. Another example is sea ice. Sea ice thermodynamic models are based on a diffusion equation very similar to (1) with a free open BC [39]. The minimum internal entropy production principle could be applied to help finding the diffusion coefficients, but only if the physics of the minimum principle is established.

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Appendix A. (Extended Euler variation with Neumann BC)

For an integral function

$$J[u(x)] = \int_L f(x, u, u', \ldots),$$

assuming a small variation to u as $u + \delta u$, where $\ldots$ represents the higher order derivatives, thus

$$\begin{aligned}\delta J[u(x)] &= \int_L [f(x, u + \delta u, u' + \delta u', \ldots) - f(x, u, u', \ldots)]\\ &\approx \int_L \left(\frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial u'}\delta u' + \cdots\right).\end{aligned} \tag{A.1}$$

In (A.1),

$$\begin{aligned}\int_L \frac{\partial f}{\partial u'}\delta u' &= \left[\frac{\partial f}{\partial u'}\delta u\right]\bigg|_{\partial L} - \int_L \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial u'}\delta u\\ &= -\int_L \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial u'}\delta u,\end{aligned} \tag{A.2}$$

where the Dirichlet BC $\delta u = 0$ $(x \in \partial L)$ is applied, and similarly for the higher order derivatives. Therefore (A.1) becomes

$$\delta J[u(x)] \approx \int_L \left(\frac{\partial f}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial u'} + \cdots \right) \delta u. \tag{A.3}$$

At the extreme point $\delta J[u(x)] = 0$. Since (A.3) is true for arbitrary δu,

$$\frac{\partial f}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial u'} + \cdots (-)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\partial f}{\partial u^{[n]}} + \cdots = 0, \tag{A.4}$$

where $[n]$ means the nth derivative. (A.4) is the traditional Euler variation equation.

Now let us consider the variation under the Neumann BC $\delta u' = 0$ $(x \in \partial L)$. Similarly to (A.1), we can derive

$$\delta J[u(x)] = \delta u \left[\int \frac{\partial f}{\partial u} \right]\Bigg|_{\partial L} + \int_L \left[-\left(\int \frac{\partial f}{\partial u} \right) + \frac{\partial f}{\partial u'} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial u''} + \cdots \right] \delta u'. \tag{A.5}$$

If u does not explicitly appear in $J[u(x)]$ (as (12)), the variation equation is

$$\frac{\partial f}{\partial u'} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial u''} + \cdots (-)^{n+1} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\partial f}{\partial u^{[n+1]}} + \cdots = 0. \tag{A.6}$$

If u does explicitly appear in $J[u(x)]$, the corresponding variation equation is difficult to obtain since the first term on the right-hand side of (A.5) is not easy to handle. Euler variation is generally presented for Dirichlet BC as (A.4). It is possible that the results for Neumann BC as (A.5) might be presented elsewhere but the author is not aware of any such publication.

Appendix B. (Solutions of steady diffusion equation)

The general solution of (3) is

$$u(x) = -\int \tilde{f}(\xi)\mathrm{sh}[\lambda(x-\xi)]\,\mathrm{d}\xi + c_1\,\mathrm{e}^{-x} + c_2\,\mathrm{e}^{x} \tag{B.1}$$

where $\lambda^2 = \gamma/k$, $\tilde{f} = f/k$, and the integral constants c_1 and c_2 are determined by the BC.

Under a Neumann BC, in the domain $L = [x_1, x_2]$, let $v_{1,2}$ be the values of u' at $x_{1,2} \in \partial L$,

$$u(x) = -\frac{1}{\lambda}\int_{x_1}^{x} \tilde{f}(\xi)\mathrm{sh}[\lambda(x-\xi)]\,\mathrm{d}\xi + \frac{(v_2+H)\mathrm{ch}[\lambda(x-x_1)] - v_1\mathrm{ch}[\lambda(x_2-x)]}{\lambda\mathrm{sh}[\lambda(x_2-x_1)]}, \tag{B.2}$$

where

$$H = \int_L \tilde{f}(\xi)\mathrm{ch}[\lambda(x_2-\xi)]\,\mathrm{d}\xi.$$

The system is closed when $v_{1,2} = 0$.

Under Dirichlet BC (free open BC), let $u_{1,2}$ be the values u at $x_{1,2} \in \partial L$,

$$u(x) = -\frac{1}{\lambda}\int_{x_1}^{x} \tilde{f}(\xi)\mathrm{sh}[\lambda(x-\xi)]\,\mathrm{d}\xi + \frac{u_1\mathrm{sh}[\lambda(x_2-x)] + (u_2+H)\mathrm{sh}[\lambda(x-x_1)]}{\mathrm{sh}[\lambda(x_2-x_1)]}, \tag{B.3}$$

where

$$H = \frac{1}{\lambda}\int_L \tilde{f}(\xi)\mathrm{sh}[\lambda(x_2-\xi)]\,\mathrm{d}\xi.$$

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